Lecture 17

Friday, 4 November 2022 1:04 AM

Revelation Principle, Myerson's Lemma, Sponsored Search Auctions.

Last time:

Vickrey's Second Price Auction.

- n agents / bidders, each had a value $v_i \in \mathbb{R}_+$ & could choose a bid $b_i \in \mathbb{R}_+$
- set of alternatives A = set of agents
- Mechanism (x, p) : $x : \mathbb{R}_+^n \to A$ (i/p: n bids)

  $p_i : \mathbb{R}_+^n \to \mathbb{R}_+$

$b = (b_1, \dots, b_n)$

then $u_i(b) = v_i - p_i$ if $x(b) = i$, i.e., agent i gets the item

$= -p_i$ o.w.

(but $p_i = 0$ if i is not allo

VSP auction: Let $i^* = \arg\max b_i$ (assume unique)

then $x(b) = i^*$, $p_{i^*} = \max_{i \neq i^*} b_i$, $p_i = 0$ for $i \neq i^*$

We saw that this mechanism is IC (truthtelling is a weakly dominant strategy), and maximizes social welfare.

Suppose our goal is to maximize welfare.

Are there other mechanisms that achieve this?

Consider the following (somewhat silly) mechanism.
Assume all v_i's are at least some large number, say 100.
Mechanism: give item to the highest bidder.
charge the winning bidder, 1+ second-highest bid.

Q. does this mechanism still maximize social welfare?
Q. Is it still IC?
Q. If not, is there still a weakly dominant strategy?

Claim: For each bidder i, bidding $b_i = v_i - 1$ is now a WDS

(prove yourself)

hence, the answers are yes, no, yes..
In particular, there is another mechanism that obtains the same "outcome" (maximizing SW), is not IC, but has a weakly dominant strategy for each player.

More generally, in cardinal mechanism design w/ money:

- set A of m alternatives
- set N of n agents, value $v_i : A \to \mathbb{R}$, $v_i \in V_i$
  bids $b_i : A \to \mathbb{R}$, $b_i \in B_i$

- a SCF $f : V_1 \times \dots \times V_n \to A$ $(B_i$ is set of all fns $A \to \mathbb{R})$
- a mechanism $M(x, p)$, $x : B_1 \times \dots \times B_n \to A$

  $p_i : B_1 \times \dots \times B_n \to \mathbb{R}_+$

  agent i's utility : $u_i : A \times \mathbb{R} \to \mathbb{R}$

  $u_i(a, p_i) = v_i(a) - p_i$
- strategy set $S_i : V_i \to B_i$

  (fix M) (we want to think of truthfully as a strategy. So a strategy is a map from values for agent i to bids)

For fixed M, the strategy profile $s = (s_1, \dots, s_n)$ is an equilibrium if $\forall i, \forall v_i$,

$$v_i(x(b_i, b_{-i})) - p_i(b_i, b_{-i}) \geq v_i(x(b_i', b_{-i})) - p_i(b_i', b_{-i})$$

where $\forall j \in N$, $b_j = s_j(v_j)$

Note that we're drawing a distinction between SCFs $f : V_1 \times \dots \times V_n \to A$, which depend on actual values, and mechanisms, that see only bids.

Thus, ① give the item to bidder w/ highest value is an SCF

(ii) give item to bidder w/ highest bid, charge xyz, is a mechanism.

A mechanism $M = (x, p)$ is IC if $\forall i, \forall v_i, \forall b_{-i}$, the strategy $s_i(v_i) = v_i$ is a WDS.

(also called Dominant Strategy Incentive Compatible)

A SCF f is "implemented" by a mechanism $M = (x, p)$ in dominant strategies if for all $i \in N$, $\exists$ strategy s_i s.t.

① s_i is a WDS, i.e., $\forall v_i, b_{-i}$,

$$u_i(x(s_i(v_i), b_{-i})) - p_i(s_i(v_i), b_{-i}) \geq u_i(x(b_i', b_{-i})) - p_i(b_i', b_{-i})$$

(II) $f(v) = x(b)$

i.e., $\exists$ WDS for all players, and if players play these strategies, the outcome is the same as f.

Note that this is weaker than asking for DSIC. We only want that $\forall v$, $\exists$ some WDS for which the mechanism gives us $f(v)$.

But in fact the two are the same. For a SCF f, if $\exists$ a mechanism that implements f in WDS, then there exists an IC mechanism that implements f.

Theorem (Revelation Principle): If an abitrary mech. implements f in WDS, then there is an IC mechanism that implements f.

Proof : Suppose $M = (x, p)$ implements f in WDS. Let s_i be the WDS for player i. Then $x(b) = f(v)$.

Consider the mechanism $M' = (x', p')$ that does the strategizing for the agents:

Then $x'(b) = x(s_1(b_1), \ldots, s_n(b_n))$

& $\forall i, p_i'(b) = p_i(s_1(b_1), \ldots, s_n(b_n))$

Note that earlier, $\forall v = (v_1, \ldots, v_n)$, $\forall i$, $\forall b_{-i}'$

since s_i is a WDS for mechanism $M = (x, p)$,

$$v_i(x(s_i(v_i), b_{-i}) - p_i(s_i(v_i), b_{-i}) \geq v_i(x(b_i', b_{-i})) - p_i(b_i', b_{-i})$$

(i) M' is DSIC :

Note that :

$x'(v_i, b_{-i}') = x(b_i, b_{-i}'')$, $p_i'(v_i, b_{-i}') = p_i(b_i, b_{-i}'')$

hence, for all i, v_i, b_{-i}',

$$\begin{aligned} x'(v_i, b_{-i}') - p_i'(v_i, b_{-i}') &= x(b_i, b_{-i}'') - p_i(b_i, b_{-i}'') \\ &\geq x(b_i', b_{-i}'') - p_i(b_i', b_{-i}'') \\ &= x(v_i', b_{-i}') - p_i(v_i', b_{-i}') \end{aligned}$$

(truthtelling is a weakly dominant strategy)

(ii) M' implements f

Fix v, let b be (weakly dominant) bids for M.

Then $x'(v) = x(b) = f(v)$ ∎

Ok, so what SCFs can be implemented by IC mechanisms?

We will look at single-parameter domains, & give a simple characterization of SCFs implementable by IC mechanisms.

Single-Parameter Domains

- $A \subseteq \mathbb{R}^n$ $(x = (x_1, \ldots, x_n)$, x_i is amt that agent i gets$)$
- for each bidder i, $B_i = V_i = \mathbb{R}_+$, i's value per unit item.
- $M = (x, p)$, $x : \mathbb{R}^n_+ \to A$, $p_i : \mathbb{R}^n_+ \to \mathbb{R}_+$

  $u_i(b) = v_i \cdot x_i(b) - p_i(b)$

  (given M)

Defn: For a single-parameter domain, SCF f is monotone if $\forall i, v_i, v_{-i}, v_i' \leq v_i$,

$$(f(v_i, v_{-i}))_i \geq (f(v_i', v_{-i}))_i$$

Theorem: For a single-parameter domain

(1) SCF f is implementable by a DSIC mechanism iff f is monotone

(2) If f is monotone, there is a unique payment rule so that the mechanism $(x = f, p)$ is DSIC (assuming $b_i = 0 \Rightarrow (f(b))_i = 0$, $p_i(b) = 0$)

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Proof: Fix i, and since we are only interested in i's bids, fix b_{-i}. Then $x_i(z) = x_i(z, b_{-i})$, $p_i(z) = p_i(z, b_{-i})$

① implementable $\Rightarrow$ monotone :

$\forall y, z \in \mathbb{R}_+$, $\quad y\, x_i(z) - p_i(z) \leq y\, x_i(y) - p_i(y)$ (since DSIC)

& $\quad z\, x_i(y) - p_i(y) \leq z\, x_i(z) - p_i(z)$

thus, $y(x_i(z) - x_i(y)) \leq p_i(z) - p_i(y) \leq z(x_i(z) - x_i(y))$

or, $(z-y)(x_i(z) - x_i(y)) \geq 0$

thus if $z > y$, $x_i(z) \geq x_i(y)$

(ii) monotone $\Rightarrow$ implementable

(we will do this for some special cases)

from above:

$$y(x_i(z) - x_i(y)) \leq p_i(z) - p_i(y) \leq z(x_i(z) - x_i(y))$$

Suppose $z > y$

$$y\,\frac{x_i(z) - x_i(y)}{z-y} \leq \frac{p_i(z) - p_i(y)}{z-y} \leq z\,\frac{x_i(z) - x_i(y)}{z-y}$$

Taking limits, and assuming $x_i(\cdot)$ is differentiable, we get

$$p_i'(z) = z\, x_i'(z) \qquad \forall z \in \mathbb{R}'$$

Together w/ $p_i(0) = 0$, this gives

$$p_i(z) = \int_{w=0}^{w=z} w\, x_i'(w)\, dw$$

This is thus a necessary condition for the mechanism to be DSIC

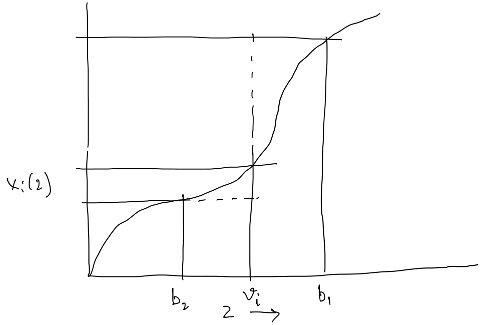
Using integration by parts,

$$p_i(z) = \left[w \int x_i'(w)\,dw - \int \int x_i'(w)\,dw\,dw \right]_{w=0}^{w=z}$$

$$= \left[w\, x_i(w) - \int x_i(w)\,dw \right]_{w=0}^{z}$$

$$= z\, x_i(z) - \int_{w=0}^{z} x_i(w)\,dw$$

So,



So if $b_i = v_i$, $u_i(b_i) = \int_0^{v} x_i(w)\,dw$

$$\left.\begin{array}{l} \text{If } b_i = b_1 > v_i, \; u_i(b_i) \leq u_i(v_i) \\ \text{if } b_i = b_2 < v_i, \; u_i(b_i) \leq u_i(v_i) \end{array}\right\} \text{see from picture!}$$

hence, if x is monotone, the mechanism $M = (x, p)$ is DSIC. ■

Claim: Suppose $x_i(z)$ is piecewise constant. i.e., ∃ breakpoints $z_1 < z_2 < z_3 < \dots < z_k$

s.t.

$$x_i(z) = \begin{cases} 0 & \text{for } z \leq z_1 \\ c_1 & \text{for } z_1 < z \leq z_2 \\ \vdots & \\ c_k & \text{for } z > z_k \end{cases}$$

and $0 \leq c_1 \leq \dots \leq c_k$

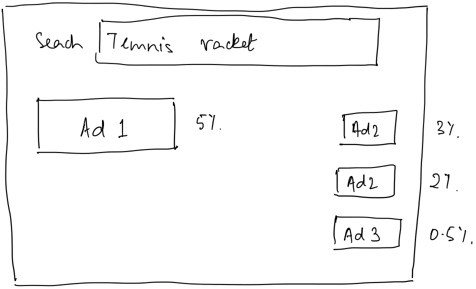
Then for the mechanism $M = (x, p)$ to be DSIC,

$$p_i(z) = \sum_{j : z_j < z} z_j (c_j - c_{j-1}), \quad \text{taking } c_0 = 0$$

(prove yourself)

Sponsored Search Auctions.

Models how ads appear on a web page



Advertisers value how many times their ad is clicked

= # of visits to their homepage

Each ad slot has a "click-through rate" CTR, which is the probability (irresp. of the ad) that a user will click on an ad in that slot.

Further, each advertiser i has a fixed value v_i per click. Thus if i gets a slot w/ CTR α_j, her value is $v_i \alpha_j$.

Say there are n advertisers, n ad slots in a page, ad slot j has CTR α_j. Say $\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_n = 0$ (some of these others may also be zero).

Our goal is to construct a DSIC mechanism that assigns slot j to the bidder that has j^{th} highest value.

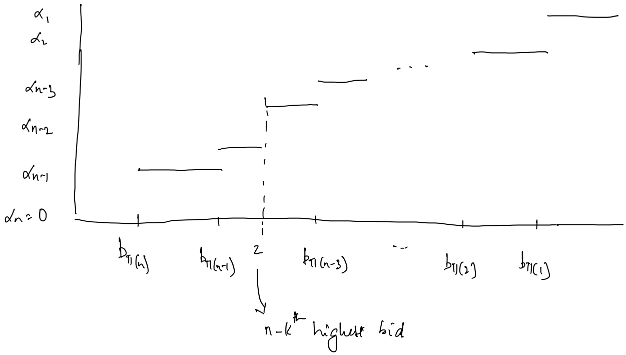
(i.e., maximize social welfare)

$A = \{\text{all permutations of } (\alpha_1, \alpha_2, \dots, \alpha_n)\}$

$V_i = B_i = \mathbb{R}_+ \quad \forall i$

Mechanism $x_i(b) = \alpha_j$, if bidder i has j^{th} highest bid.

This is monotone, hence implementable as a DSIC mechanism.



say there are $k-1$ bids $< z$

$$b_{\pi(n)}, \ldots, b_{\pi(n-k+1)}$$

then $$p_i(z) = \sum_{j=1}^{k-1} (\alpha_{n-j} - \alpha_{n-(j-1)}) \, b_{\pi(n-j)}$$

In practice, this is too complicated. What is used is the "generalized second price" auction, where ith highest bidder pays the $(i+1)$th highest bid. This is not DSIC, but "works in practice".